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Approximations of Two-Place Functions by Finite Sums of Terms with Separable Variables

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It is known that if a smooth function h in two real variables x and y belongs to the class Σ_n of all sums of the form $\sum_{k=1}^{n} f_k(x)\, g_k(y)$, then its $(n+1)$th order "Wronskian" $\det[h_{x^i y^j}]_{i,j=0}^{n}$ is identically equal to zero. The present paper deals with the approximation problem $h(x, y) \approx \sum_{k=1}^{n} f_k(x)\, g_k(y)$ with a prescribed n, for general smooth functions h not lying in Σ_n. Two natural approximation sums $T = T(h) \in \Sigma_n$, $S = S(h) \in \Sigma_n$ are introduced and the errors $|h - T|$, $|h - S|$ are estimated by means of the above mentioned Wronskian of the function h. The proofs utilize the technique of ordinary linear differential equations.

1. INTRODUCTION. STATEMENT OF THE PROBLEM

Scalar functions of the form

$$\sum_{k=1}^{n} f_k(x)\, g_k(y) \tag{1}$$

arise in many areas of pure and applied mathematics (see [3]). C. M. Stéphanos [9] was probably the first who introduced the Wronski-like matrices built from the partial derivatives of a given two-place function $h = h(x, y)$

$$W_n = W_n(h) := \begin{pmatrix} h & h_y & \cdots & h_{y^{n-1}} \\ h_x & h_{xy} & \cdots & h_{xy^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ h_{x^{n-1}} & h_{x^{n-1}y} & \cdots & h_{x^{n-1}y^{n-1}} \end{pmatrix} \qquad (n = 1, 2, \ldots) \tag{2}$$

and who observed that each (smooth) function h representable by a sum (1) has to satisfy the equation $\det W_{n+1} \equiv 0$. A more precise and complete assertion given by Neuman in [5] can be stated as the following

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Theorem A. Throughout the paper, $n = 1, 2, \ldots$ is a fixed integer, $I = [a, b]$ and $J = [c, d]$ are two compact intervals in $\mathbb{R}$ and the function $h\colon I \times J \to \mathbb{R}$ is assumed to have the partial derivative $h_{x^n y^n}$ continuous on $I \times J$.

THEOREM A. *If h is of the form* (1), *then* $\det W_{n+1} \equiv 0$ *on* $I \times J$. *Conversely, if*

$$\det W_n(x, y) \neq 0 \tag{3}$$

and

$$\det W_{n+1}(x, y) = 0 \qquad \text{at each point } (x, y) \in I \times J,$$

then h is of the form (1), *with linearly independent components f_k and g_k.*

Proof. see [5, Thm. 1] or [3, Thm. 3.1].

Remark 1. Rassias [7] gave a counter-example showing that the Stéphanos condition $\det W_{n+1} \equiv 0$ is not sufficient for h to be of the form (1) if there exist some zeros of $\det W_n$ (but not $\det W_n \equiv 0$). As a by-product of our considerations, we establish a new differential criterion for h to be of type (1), applicable just in the case when $\det W_n$ has at least one non-zero value in the rectangle $I \times J$ (Theorem 3 in Section 3).

Remark 2. A non-trivial extension of Theorem A to the case of multi-dimensional x and y was established in [2]. For the functions of more than two variables, the decompositions like (1) were discussed in [1, 6].

This paper is concerned with the approximation problem

$$h(x, y) \approx \sum_{k=1}^{n} f_k(x)\, g_k(y) \tag{4}$$

which seems to be of interest for each function h not permitting any exact representation (1). We start our procedure by introducing two natural approximating sums $T = T(h)$ and $S = S(h)$, analogous to Taylor series and the interpolation polynomials, respectively (see Section 2). As shown in Section 3, the corresponding errors $h - T$ and $h - S$ can be represented by formulas that resemble the Lagrange form for errors in polynomial approximations. The bounds for the errors $|h - T|$ and $|h - S|$ of different types are stated in Section 4. The proofs of a part of them are postponed to Section 6, because they need fine estimates for some Wronski-like determinants discussed in Section 5. The result proved in Section 6 will lead to the conclusion that the condition (3) of Theorem A is "stable" in the following sense: If the approximated function h satisfies

$$\det W_n(x, y) \neq 0 \tag{5}$$

and

$$\left|\frac{\det W_{n+1}(x, y)}{\det W_n(x, y)}\right| \leqslant \varepsilon \qquad \text{for each } (x, y) \in I \times J,$$

with some "small" constant ε, then the sup-norms (supremum norms) of the errors $h - T$ and $h - S$ are of order ε, too.

Remark 3. While the problem of the best L^2-approximation (4), with a prescribed number n of products of arbitrary functions f_k and g_k, has been recently solved in [8], the problem of the best approximation (4) with respect to the sup-norm seems to be still open.

2. Two Types of Approximating Sums

In this section, we deal with the problem of creating suitable approximations (4). As usual on other occasions, we determine such approximations by imposing some *coincidence conditions*. In this way, we introduce two natural (and as the paper shows, also effective) approximations T and S (see (7) and (12) below).

(i) Suppose that $\det W_n(x_0, y_0) \neq 0$ for some fixed $x_0 \in I$ and $y_0 \in J$. Let us show that the (unique) function $T: I \times J \to \mathbb{R}$ of the form (1) that satisfies $2n$ functional conditions

$$T_{x^j}(x_0, \cdot) = h_{x^j}(x_0, \cdot)$$

and

$$T_{y^j}(\cdot, y_0) = h_{y^j}(\cdot, y_0) \qquad (0 \leqslant j \leqslant n-1) \tag{6}$$

can be written as a matrix product

$$T(x, y) = (h(x, y_0), h_y(x, y_0), ..., h_{y^{n-1}}(x, y_0)) \cdot W_n^{-1}(x_0, y_0) \cdot \begin{pmatrix} h(x_0, y) \\ h_x(x_0, y) \\ \vdots \\ h_{x^{n-1}}(x_0, y) \end{pmatrix}, \tag{7}$$

where W_n^{-1} denotes the inverse of the matrix W_n. Indeed, if T is as in (1) and satisfies (6), then

$$h_{x^j}(x_0, \cdot) = \sum_{k=1}^{n} f_k^{(j)}(x_0)\, g_k \tag{8}$$

and

$$h_{y^j}(\cdot, y_0) = \sum_{k=1}^{n} g_k^{(j)}(y_0) f_k \qquad (0 \leqslant j \leqslant n-1).$$

We may consider (8) as two linear algebraic systems with unknowns $g_1, g_2, ..., g_n$ and $f_1, f_2, ..., f_n$, respectively. Note that the matrices of these systems

$$F(x_0) = \begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f'_1(x_0) & f'_2(x_0) & \cdots & f'_n(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix}$$

and

$$G(y_0) = \begin{pmatrix} g_1(y_0) & g_2(y_0) & \cdots & g_n(y_0) \\ g'_1(y_0) & g'_2(y_0) & \cdots & g'_n(y_0) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n-1)}(y_0) & g_2^{(n-1)}(x_0) & \cdots & g_n^{(n-1)}(y_0) \end{pmatrix}$$

are non-singular, because (8) implies that

$$F(x_0) \cdot G^T(y_0) = W_n(x_0, y_0) \tag{9}$$

and $W_n(x_0, y_0)$ is supposed to be non-singular. Hence (8) yields

$$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} = F^{-1}(x_0) \cdot \begin{pmatrix} h(x_0, \cdot) \\ h_x(x_0, \cdot) \\ \vdots \\ h_{x^{n-1}}(x_0, \cdot) \end{pmatrix}$$

and

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = G^{-1}(y_0) \cdot \begin{pmatrix} h(\cdot, y_0) \\ h_y(\cdot, y_0) \\ \vdots \\ h_{y^{n-1}}(\cdot, y_0) \end{pmatrix}.$$

Substituting this into $T = \sum_{k=1}^{n} f_k g_k$ and taking in account (9), we conclude that (7) holds. On the other side, it is easy to check that the function T defined by (7) satisfies (6).

(ii) Suppose that $x_1, x_2, ..., x_n \in I$ and $y_1, y_2, ..., y_n \in J$ are chosen so that the matrix

$$H := \begin{pmatrix} h(x_1, y_1) & h(x_1, y_2) & \cdots & h(x_1, y_n) \\ h(x_2, y_1) & h(x_2, y_2) & \cdots & h(x_2, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ h(x_n, y_1) & h(x_n, y_2) & \cdots & h(x_n, y_n) \end{pmatrix} \tag{10}$$

is non-singular. Let us show that the (unique) function $S: I \times J \to \mathbb{R}$ of the form (1) satisfying $2n$ functional conditions

$$S(x_j, \cdot) = h(x_j, \cdot) \qquad \text{and} \qquad S(\cdot, y_j) = h(\cdot, y_j) \qquad (1 \leqslant j \leqslant n) \tag{11}$$

can be written as a matrix product

$$S(x, y) = (h(x, y_1), h(x, y_2), ..., h(x, y_n)) \cdot H^{-1} \cdot \begin{pmatrix} h(x_1, y) \\ h(x_2, y) \\ \vdots \\ h(x_n, y) \end{pmatrix}. \tag{12}$$

We can proceed analogously as in part (i). If S is as in (1) and satisfies (11), then we can compute $g_1, g_2, ..., g_n$ and $f_1, f_2, ..., f_n$ from the systems

$$h(x_j, \cdot) = \sum_{k=1}^{n} f_k(x_j)\, g_k \qquad \text{and} \qquad h(\cdot, y_j) = \sum_{k=1}^{n} g_k(y_j)\, f_k \qquad (1 \leqslant j \leqslant n)$$

and conclude that (12) holds. Conversely, S from (12) obviously satisfies (11).

Remark 3. Neuman [5] showed that if the matrix H from (10) is singular for each $x_1, x_2, ..., x_n \in I$ and each $y_1, y_2, ..., y_n \in J$, then h is of the form

$$h(x, y) = \sum_{k=1}^{n'} f_k(x)\, g_k(y) \qquad \text{for some } n' < n.$$

Consequently, for each function h not permitting any exact representation (1), we can choose $x_1, x_2, ..., x_n \in I$ and $y_1, y_2, ..., y_n \in J$ so that the sum S in (12) is well-defined.

3. Error Representation

After introducing the approximating sums T and S in Section 2, we now turn our attention to the problem of representation of the errors $h - T$ and $h - S$.

Recall first the well-known Lagrange formulas

$$z(t_0)=z'(t_0)=\cdots=z^{(n-1)}(t_0)=0 \Rightarrow z(t)=\frac{z^{(n)}(\xi)}{n!}\cdot(t-t_0)^n \tag{13}$$

and

$$z(t_1)=z(t_2)=\cdots=z(t_n)=0 \Rightarrow z(t)=\frac{z^{(n)}(\xi)}{n!}\cdot\prod_{i=1}^{n}(t-t_i), \tag{14}$$

where $t_0, t_1, ..., t_n, t, \xi \in I$ and $t_i \neq t_j$ $(1 \leqslant i < j \leqslant n)$, being valid for each function z possessing the nth derivative on the interval I. To state our Lagrange-like formulas for the errors $h-T$ and $h-S$ in approximation (4), it is convenient to introduce the following "remainder" determinants

$$d_T(x, y) := \det\begin{pmatrix} h(x_0, y_0) & h_y(x_0, y_0) & \cdots & h_{y^{n-1}}(x_0, y_0) & h_{y^n}(x_0, y) \\ h_x(x_0, y_0) & h_{xy}(x_0, y_0) & \cdots & h_{xy^{n-1}}(x_0, y_0) & h_{xy^n}(x_0, y) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{x^{n-1}}(x_0, y_0) & h_{x^{n-1}y}(x_0, y_0) & \cdots & h_{x^{n-1}y^{n-1}}(x_0, y_0) & h_{x^{n-1}y^n}(x_0, y) \\ h_{x^n}(x, y_0) & h_{x^n y}(x, y_0) & \cdots & h_{x^n y^{n-1}}(x, y_0) & h_{x^n y^n}(x, y) \end{pmatrix} \tag{15}$$

and

$$d_S(x, y) := \det\begin{pmatrix} h(x_1, y_1) & h(x_1, y_2) & \cdots & h(x_1, y_n) & h_{y^n}(x_1, y) \\ h(x_2, y_1) & h(x_2, y_2) & \cdots & h(x_2, y_n) & h_{y^n}(x_2, y) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(x_n, y_1) & h(x_n, y_2) & \cdots & h(x_n, y_n) & h_{y^n}(x_n, y) \\ h_{x^n}(x, y_1) & h_{x^n}(x, y_2) & \cdots & h_{x^n}(x, y_n) & h_{x^n y^n}(x, y) \end{pmatrix}. \tag{16}$$

THEOREM 1. *Let* $x_0 \in I$ *and* $y_0 \in J$ *be chosen so that the matrix* $W_n(x_0, y_0)$ *is non-singular. Define* T *and* d_T *by* (7) *and* (15), *respectively. Then for each* $x \in I$ *and* $y \in J$, *the difference* $h(x, y) - T(x, y)$ *can be represented as*

$$h(x, y) - T(x, y) = \frac{d_T(\xi, \eta)}{(n!)^2 \det W_n(x_0, y_0)} \cdot (x-x_0)^n\,(y-y_0)^n, \tag{17}$$

where $\xi = \xi(x, y)$ *lies between* x_0 *and* x, *while* $\eta = \eta(x, y)$ *lies between* y_0 *and* y.

Proof. In view of (6), the function $\lambda := h - T$ satisfies

$$\lambda_{x^j}(x_0, \cdot) = 0 \qquad \text{and} \qquad \lambda_{y^j}(\cdot, y_0) = 0 \qquad (0 \leqslant j \leqslant n-1). \tag{18}$$

Thus we can apply (13) to the function $z=\lambda(\cdot, y)$ with a fixed $y \in J$ and conclude that

$$\lambda(x, y)=\frac{\lambda_{x^n}(\xi, y)}{n!}\cdot(x-x_0)^n, \tag{19}$$

where $\xi=\xi(x, y)$ lies between x_0 and x. It follows from the second part of (18) that $\lambda_{x^n y^j}(\cdot, y_0)=0$, for each $0 \leqslant j \leqslant n-1$. Now applying (13) to the function $z=\lambda_{x^n}(\xi, \cdot)$ with a fixed $\xi \in I$, we obtain

$$\lambda_{x^n}(\xi, y)=\frac{\lambda_{x^n y^n}(\xi, \eta)}{n!}\cdot(y-y_0)^n, \tag{20}$$

where $\eta=\eta(\xi, y)$ lies between y_0 and y. Substituting (20) into (19), we observe that (17) holds if

$$\frac{d_T(\xi, \eta)}{\det W_n(x_0, y_0)}=h_{x^n y^n}(\xi, \eta)-T_{x^n y^n}(\xi, \eta). \tag{21}$$

However, the last equality follows from definitions (7), (15) and from an elementary proposition of matrix theory,

If $A_n=[a_{ij}]_{i,j=1}^{n}$ *and* $A_{n+1}=[a_{ij}]_{i,j=1}^{n+1}$ *and if* $\det A_n \neq 0$, *then*

$$\frac{\det A_{n+1}}{\det A_n}=a_{n+1,n+1}-(a_{1,n+1}, ..., a_{n,n+1})\cdot A_n^{-1}\cdot\begin{pmatrix} a_{n+1,1} \\ \vdots \\ a_{n+1,n}\end{pmatrix}. \tag{22}$$

Thus the proof is complete.

COROLLARY 1. *Let* $x_0 \in I$ *and* $y_0 \in J$ *be chosen so that the matrix* $W_n(x_0, y_0)$ *is non-singular. Define* T *by* (7). *Then the following asymptotic formula holds*

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{h(x, y)-T(x, y)}{(x-x_0)^n\,(y-y_0)^n}=\frac{\det W_{n+1}(x_0, y_0)}{(n!)^2\cdot\det W_n(x_0, y_0)}. \tag{23}$$

Proof. If $x \to x_0$ and $y \to y_0$ in (17), then $\xi \to x_0$, $\eta \to y_0$ and therefore by continuity, $d_T(\xi, \eta) \to d_T(x_0 \cdot y_0)=\det W_{n+1}(x_0, y_0)$, which proves (23).

THEOREM 2. *Let* $x_1, x_2, ..., x_n \in I$ *and* $y_1, y_2, ..., y_n \in J$ *be chosen so that the matrix* H *in* (10) *is non-singular. Define* S *and by* d_S *by* (12) *and* (16), *respectively. Then for each* $x \in I$ *and* $y \in J$, *the difference* $h(x, y)-S(x, y)$ *can be represented as*

$$h(x, y)-S(x, y)=\frac{d_S(\xi, \eta)}{(n!)^2 \det H}\cdot\prod_{k=1}^{n}(x-x_k)(y-y_k), \tag{24}$$

where $\xi=\xi(x, y)$ and $\eta=\eta(x, y)$ lie in the minimal subintervals of I and J containing all the points $x, x_1, \ldots, x_n$ and $y, y_1, \ldots, y_n$, respectively.

Proof. In view of (11), the function $\mu := h - S$ satisfies

$$\mu(x_j, \cdot)=0 \qquad \text{and} \qquad \mu(\cdot, y_j)=0 \qquad (1 \leqslant j \leqslant n). \tag{25}$$

Thus we can apply (14) to the function $z=\mu(\cdot, y)$ with a fixed $y \in J$ and conclude that

$$\mu(x, y)=\frac{\mu_{x^n}(\xi, y)}{n!} \cdot \prod_{k=1}^{n} (x-x_k),$$

where $\xi=\xi(x, y)$ lies in the subinterval of I mentioned above. It follows from the second part of (25) that $\mu_{x^n}(\cdot, y_j)=0$, for each $1 \leqslant j \leqslant n$. Now applying (14) to the function $z=\mu_{x^n}(\xi, \cdot)$ with a fixed $\xi \in I$, we obtain

$$\mu_{x^n}(\xi, y)=\frac{\mu_{x^n y^n}(\xi, \eta)}{n!} \cdot \prod_{k=1}^{n} (y-y_k),$$

where $\eta=\eta(\xi, y)$ lies in the subinterval of J mentioned above. Consequently, (24) holds if

$$\frac{d_S(\xi, \eta)}{\det H}=h_{x^n y^n}(\xi, \eta)-S_{x^n y^n}(\xi, \eta). \tag{26}$$

However, the last equality follows from definitions (12), (16) and from the rule (22).

The proof is complete.

Remark 3. The reader might feel the lack of some convergence formula like (23) which could illustrate the asymptotic behavior of the error $h-S$. It is due to the fact that we cannot arrange the behavior of x and y so that the "unknowns" ξ and η in (24) may converge to some definite limits. Nevertheless, one can propose for example, the problem of determining n^2 limits

$$\lim_{\substack{x \to x_p \\ y \to y_q}} \frac{h(x, y)-S(x, y)}{(x-x_p)(y-y_q)}, \qquad p, q \in \{1, 2, \ldots, n\}.$$

We are in doubt whether any "reasonable" formulas for these limits exist at all.

Let us finish this section by proving a new criterion for decompositions (1) mentioned above in Remark 1.

THEOREM 3. *Let $x_0 \in I$ and $y_0 \in J$ be chosen so that the matrix $W_n(x_0, y_0)$ is non-singular. Then the function h is of the form* (1), *with some components $f_k \in C^n(I)$ and $g_k \in C^n(J)$, if and only if the determinant d_T from* (15) *satisfies $d_T \equiv 0$ on $I \times J$.*

Proof. If h is as in (1), then the $(n+1)\times(n+1)$-matrix from (15) can be written as the product of the $(n+1)\times n$-matrix

$$\begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f'_1(x_0) & f'_2(x_0) & \cdots & f'_n(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \\ f_1^{(n)}(x) & f_2^{(n)}(x) & \cdots & f_n^{(n)}(x) \end{pmatrix}$$

times the $n \times (n+1)$-matrix

$$\begin{pmatrix} g_1(y_0) & g'_1(y_0) & \cdots & g_1^{(n-1)}(y_0) & g_1^{(n)}(y) \\ g_2(y_0) & g'_2(y_0) & \cdots & g_2^{(n-1)}(y_0) & g_2^{(n)}(y) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_n(y_0) & g'_n(y_0) & \cdots & g_n^{(n-1)}(y_0) & g_n^{(n)}(y) \end{pmatrix}.$$

Since the rank of such a product does not exceed n, the common size of both the factors, we have $d_T \equiv 0$. Conversely, if $d_T \equiv 0$, then Theorem 1 implies that $h \equiv T$. Since T is of type (1), the proof is complete.

Remark 4. Using Theorem 2, one can clearly obtain another new criterion for decompositions (1) in the form $d_S \equiv 0$. However, this result seems to have no essential adventage in comparison with a result of Neuman [5]: *If the matrix H in* (10) *is non-singular, then h is of the form* (1) *if and only if the determinant*

$$\det \begin{pmatrix} h(x_1, y_1) & h(x_1, y_2) & \cdots & h(x_1, y_n) & h(x_1, y) \\ h(x_2, y_1) & h(x_2, y_2) & \cdots & h(x_2, y_n) & h(x_2, y) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(x_n, y_1) & h(x_n, y_2) & \cdots & h(x_n, y_n) & h(x_n, y) \\ h(x, y_1) & h(x, y_2) & \cdots & h(x, y_n) & h(x, y) \end{pmatrix}$$

vanishes for each $(x, y) \in I \times J$, which does not even require any smoothness restriction on the function h. (Compare the last determinant with that from (16).)

4. Error Estimation

It is clear from the coincidence conditions (6) and (11) that the most effective estimates of the errors $|h-T|$ and $|h-S|$ we may expect should be of the form

$$|h(x, y)-T(x, y)| \leqslant \alpha_T(x, y)\, |(x-x_0)(y-y_0)|^n \qquad (x \in I,\ y \in J) \tag{27}$$

and

$$|h(x, y)-S(x, y)| \leqslant \alpha_S(x, y) \prod_{k=1}^{n} |(x-x_k)(y-y_k)| \qquad (x \in I,\ y \in J), \tag{28}$$

where α_T and α_S are some bounded functions (or even constants). In fact, the Lagrange-like formulas (17) and (24) ensure that (27) and (28) are valid with

$$\alpha_T(x, y)=\frac{\sup\{|d_T(\xi, \eta)| : \xi \in I(x_0, x), \eta \in J(y_0, y)\}}{(n!)^2 \cdot |\det W_n(x_0, y_0)|} \tag{29}$$

and

$$\alpha_S(x, y)=\frac{\sup\{|d_S(\xi, \eta)| : \xi \in I(x_1, ..., x_n, x), \eta \in J(y_1, ..., y_n, y)\}}{(n!)^2 \cdot |\det H|} \tag{30}$$

respectively, where $I(x_1, ..., x_n, x)$ and $J(y_1, ..., y_n, y)$ denote the subintervals mentioned in the statement of Theorem 2. Let us emphasize that the estimates (27) and (28) with the factors (29) and (30) are applicable whenever the sums T and S in (7) and (12) are well-defined (without any supplementary restriction like (5) on the approximated function h). Moreover, for the sake of numerical applications, the sup-norms in (29) and (30) can be easily estimated by using the sup-norms of all the functions that occur in the determinants (15) and (16)—see Corollary 2 below.

In what follows, we use the symbol $\|\cdot\|$ to denote the sup-norm of various matrices, scalar-, and matrix-valued, one- and two-place functions. To avoid any confusion, we now list a few examples:

$$\|A\| = \max\{|a_{ij}| : 1 \leqslant i, j \leqslant n\} \text{ if } A \text{ is a constant matrix } [a_{ij}]_{i,j=1}^n,$$
$$\|h(\cdot, \cdot)\| = \sup\{|h(x, y)| : x \in I \text{ and } y \in J\},$$
$$\|h(\cdot, y)\| = \sup\{|h(x, y)| : x \in I\},$$
$$\|A(x, \cdot)\| = \sup\{\|A(x, y)\| : y \in J\},$$

etc.

COROLLARY 2. (i) *Let $x_0 \in I$ and $y_0 \in J$ be chosen so that the matrix $W_n(x_0, y_0)$ is nonsingular. Define T by* (7). *Then the estimate* (27) *holds with a constant factor α_T equal to*

$$\alpha_T = \frac{\|h_{x^ny^n}(\cdot,\cdot)\| + \|W_n^{-1}(x_0, y_0)\| \sum_{i=0}^{n-1} \|h_{x^ny^i}(\cdot, y_0)\| \sum_{j=0}^{n-1} \|h_{x^jy^n}(x_0,\cdot)\|}{(n!)^2}. \tag{31}$$

(ii) *Let $x_1, x_2, ..., x_n \in I$ and $y_1, y_2, ..., y_n \in J$ be chosen so that the matrix H in* (10) *is non-singular. Define S by* (12). *Then the estimate* (28) *holds with a constant factor α_S equal to*

$$\alpha_S = \frac{\|h_{x^ny^n}(\cdot,\cdot)\| + \|H^{-1}\| \sum_{i=1}^{n} \|h_{x^n}(\cdot, y_i)\| \sum_{j=1}^{n} \|h_{y^n}(x_j,\cdot)\|}{(n!)^2}. \tag{32}$$

Proof. Part (i) immediately follows from formulas (17) and (21), where $T_{x^ny^n}$ is computed from (7). Similarly, part (ii) follows from (24) and (26), where $S_{x^ny^n}$ is computed from (12). The proof is complete.

Although the bounds (31) and (32) are available for numerical calculations, their disadvantage (with respect to the theory of Wronski matrices (2)) is evident: *The values of* (31) *and* (32) *are non-zero in general even if the function h satisfies condition* (3) *of Theorem* A (then of course, $h \equiv T \equiv S$). This circumstance leads to the following question (in our opinion, an important one): Is it possible to derive any other estimates of type (27) and (28), more "responsive" to the condition (3)? Starting from the representation formulas (17), (23), and (24), it seems to be natural to seek some bounds for d_T and d_S depending on the sup-norm of the ratio $\det W_{n+1}/\det W_n$. Before we state our results in this direction, let us emphasize that we need an essential restriction on the function h. Namely, we will assume that the matrix W_n is non-singular *at each point* of the rectangle $I \times J$ (see the first part of condition (5)).

THEOREM 4. *Suppose that h satisfies* (5) *for some constant $\varepsilon > 0$. Choose $x_0 \in I$, $y_0 \in J$ and define T by* (7). *Then the estimate* (27) *holds with*

$$\alpha_T(x, y) = \frac{\varepsilon}{(n!)^2} \cdot (1 + K_1 |x - x_0|)(1 + K_2 |y - y_0|), \tag{33}$$

where the constants K_1, K_2 are defined by

$$\begin{aligned} K_1 &= \|W_n^{-1}(\cdot,\cdot)\| \sum_{i=0}^{n-1} \|h_{x^ny^i}(\cdot,\cdot)\| \text{ and} \\ K_2 &= \|W_n^{-1}(\cdot,\cdot)\| \sum_{j=0}^{n-1} \|h_{x^jy^n}(\cdot,\cdot)\|. \end{aligned} \tag{34}$$

Proof. see Section 6.

To prove the expected approximation property of the sum (12), we need (besides (5)) another supplementary assumption imposed on the Wronski-like matrices

$$W_{x_1, \dots, x_n}(y) := \begin{pmatrix} h(x_1, y) & h_y(x_1, y) & \cdots & h_{y^{n-1}}(x_1, y) \\ h(x_2, y) & h_y(x_2, y) & \cdots & h_{y^{n-1}}(x_2, y) \\ \vdots & \vdots & \ddots & \vdots \\ h(x_n, y) & h_y(x_n, y) & \cdots & h_{y^{n-1}}(x_n, y) \end{pmatrix}$$

and

$$W_{y_1, \dots, y_n}(y) := \begin{pmatrix} h(x, y_1) & h(x, y_2) & \cdots & h(x, y_n) \\ h_x(x, y_1) & h_x(x, y_2) & \cdots & h_x(x, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_{x^{n-1}}(x, y_1) & h_{x^{n-1}}(x, y_2) & \cdots & h_{x^{n-1}}(x, y_n) \end{pmatrix}.$$

Namely, we will assume that at least one of these matrices is non-singular at each point of the corresponding interval I or J, respectively. It is clearly no loss of generality to assume that

$$\det W_{x_1, \dots, x_n}(y) \neq 0 \text{ for each } y \in J. \tag{35}$$

THEOREM 5. *Suppose that h satisfies* (5) *for some constant $\varepsilon > 0$. Suppose also that $x_1, x_2, \dots, x_n \in I$ and $y_1, y_2, \dots, y_n \in J$ are chosen so that the matrix H in* (10) *is non-singular and that* (35) *holds. Denote by $l_1(x)$ and $l_2(y)$ the lengths of the minimal subintervals containing all the points $x, x_1, x_2, \dots, x_n$ and $y, y_1, y_2, \dots, y_n$, respectively. Define S by* (12). *Then the estimate* (28) *holds with*

$$\alpha_S(x, y) = \frac{\varepsilon}{(n!)^2} \cdot (1 + K_3 l_1(x))(1 + K_4 l_2(y)), \tag{36}$$

where the constants K_3, K_4 are defined by

$$\begin{aligned} K_3 &= \| W_n^{-1}(\cdot, \cdot)\| \, (1 + n^2 \, \| W_{x_1, \dots, x_n}^{-1}(\cdot)\| \cdot \| W_n(\cdot, \cdot)\|) \\ &\quad \times \sum_{i=0}^{n-1} \|h_{x^n y^i}(\cdot, \cdot)\| \\ K_4 &= \| W_{x_1, \dots, x_n}^{-1}(\cdot)\| \, (1 + n^2 \, \|H^{-1}\| \cdot \| W_{x_1, \dots, x_n}(\cdot)\|) \\ &\quad \times \sum_{j=1}^{n} \|h_{y^n}(x_j, \cdot)\|. \end{aligned} \tag{37}$$

Proof. see Section 6.

Remark 5. Asymmetry of the constants in (37) is due to the asymmetric condition (35).

5. Bounds for Functional Determinants

In order to make the main idea of our proofs of Theorems 4 and 5 more readable, we now separately discuss some needed estimates for determinants that depend on a system of functions in one variable.

Throughout this Section, let $z_1, ..., z_n \in C^n(I)$ be a fixed n-tuple of scalar functions such that their Wronski matrix

$$W(t) := \begin{pmatrix} z_1(t) & z_2(t) & \cdots & z_n(t) \\ z_1'(t) & z_2'(t) & \cdots & z_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(n-1)}(t) & z_2^{(n-1)}(t) & \cdots & z_n^{(n-1)}(t) \end{pmatrix}$$

is non-singular at each point t of the interval $I = [a, b]$. Suppose also that the points $t_0, t_1, ..., t_n \in I$ are fixed and that the (constant) matrix

$$Z := \begin{pmatrix} z_1(t_1) & z_2(t_1) & \cdots & z_n(t_1) \\ z_1(t_2) & z_2(t_2) & \cdots & z_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ z_1(t_n) & z_2(t_n) & \cdots & z_n(t_n) \end{pmatrix}$$

is non-singular. Put $w(t) := \det W(t)$ and define

$$\varphi(t) := \frac{1}{w(t)} \cdot \det \begin{pmatrix} z_1(t) & z_2(t) & \cdots & z_n(t) & z(t) \\ z_1'(t) & z_2'(t) & \cdots & z_n'(t) & z'(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1^{(n)}(t) & z_2^{(n)}(t) & \cdots & z_n^{(n)}(t) & z^{(n)}(t) \end{pmatrix}, \tag{38}$$

$$\psi(t) := \frac{1}{w(t_0)}$$

$$\cdot \det \begin{pmatrix} z_1(t_0) & z_2(t_0) & \cdots & z_n(t_0) & z(t_0) \\ z_1'(t_0) & z_2'(t_0) & \cdots & z_n'(t) & z'(t_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1^{(n-1)}(t_0) & z_2^{(n-1)}(t_0) & \cdots & z_n^{(n-1)}(t_0) & z^{(n-1)}(t_0) \\ z_1^{(n)}(t) & z_2^{(n)}(t) & \cdots & z_n^{(n)}(t) & z^{(n)}(t) \end{pmatrix}, \tag{39}$$

and

$$\theta(t) := \frac{1}{\det Z} \cdot \det \begin{pmatrix} z_1(t_1) & z_2(t_1) & \cdots & z_n(t_1) & z(t_1) \\ z_1(t_2) & z_2(t_2) & \cdots & z_n(t_2) & z(t_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1(t_n) & z_2(t_n) & \cdots & z_n(t_n) & z(t_n) \\ z_1^{(n)}(t) & z_2^{(n)}(t) & \cdots & z_n^{(n)}(t) & z^{(n)}(t) \end{pmatrix}, \tag{40}$$

for a given function $z \in C^n(I)$. The aim of our considerations is to estimate the values of ψ and θ by means of the sup-norm of φ.

Note that (38) can be considered as a linear non-homogeneous differential equation of order n with respect to the "unknown" z. The well-known method of variation of parameters shows that each solution z can be written in the form

$$z(t) = (z_1(t), ..., z_n(t)) \cdot \left(c + \int_{t_0}^{t} c_n W^{-1}(s)\, \varphi(s)\, ds \right), \tag{41}$$

where $c_n W^{-1}$ denotes the nth column of the matrix W^{-1} and $c \in \mathbb{R}^n$ is arbitrary. In view of a basic determinant property, it is easily seen from (39) and (40) that the functions ψ and θ do not depend on the choice of the vector c in (41). To calculate ψ, it is convenient to put $c = 0$. In fact, the function

$$z_0(t) := (z_1(t), ..., z_n(t)) \cdot \int_{t_0}^{t} c_n W^{-1}(s)\, \varphi(s)\, ds \tag{42}$$

satisfies

$$z_0^{(j)}(t) = (z_1^{(j)}(t), ..., z_n^{(j)}(t)) \cdot \int_{t_0}^{t} c_n W^{-1}(s)\, \varphi(s)\, ds \qquad (1 \leqslant j \leqslant n-1)$$

and

$$z_0^{(n)}(t) = \varphi(t) + (z_1^{(n)}(t), ..., z_n^{(n)}(t)) \cdot \int_{t_0}^{t} c_n W^{-1}(s)\, \varphi(s)\, ds, \tag{43}$$

hence $z_0(t_0) = z_0'(t_0) = \cdots = z_0^{(n-1)}(t_0) = 0$. Substituting $z = z_0$ into (39), we therefore conclude that $\psi(t) = z_0^{(n)}(t)$, for each $t \in I$. Consequently, formula (43) immediately yields the following.

LEMMA 1. *The inequality*

$$|\psi(t)| \leqslant \|\varphi(\cdot)\| \left(1 + \|W^{-1}(\cdot)\| \sum_{k=1}^{n} |z_k^{(n)}(t)| \cdot |t - t_0| \right)$$

holds for each $t \in I$.

Proof. see above.

To estimate the function θ, we first utilize the rule (22) and observe that

$$\theta(t) = z^{(n)}(t) - (z_1^{(n)}(t), \ldots, z_n^{(n)}(t)) \cdot Z^{-1} \cdot \begin{pmatrix} z(t_1) \\ z(t_2) \\ \vdots \\ z(t_n) \end{pmatrix}. \tag{44}$$

Put here $z = z_0$ again, where z_0 is defined by (42) with t_0 replaced by any t_j, say $t_0 = t_1$. In view of (42) and (43), we have

$$|z_0(t_j)| \leqslant n \, \|\varphi(\cdot)\| \cdot \|W^{-1}(\cdot)\| \cdot \|W(\cdot)\| \cdot |t_j - t_1| \qquad (j = 1, \ldots, n)$$

and

$$|z_0^{(n)}(t)| \leqslant \|\varphi(\cdot)\| \left(1 + \|W^{-1}(\cdot)\| \sum_{k=1}^{n} |z_k^{(n)}(t)| \cdot |t - t_1|\right) \qquad (t \in I).$$

Consequently, formula (44) with $z = z_0$ yields the following.

LEMMA 2. *Denote by $l(t)$ the length of the minimal subinterval containing all the points $t, t_1, t_2, \ldots, t_n$. The inequality*

$$|\theta(t)| \leqslant \|\varphi(\cdot)\| \left(1 + \|W^{-1}(\cdot)\| \, (1 + n^2 \, \|Z^{-1}\| \cdot \|W(\cdot)\|) \sum_{k=1}^{n} |z_k^{(n)}(t)| \, l(t)\right)$$

holds for each $t \in I$.

Proof. see above.

6. Proofs of Theorems 4 and 5

Now we are in position to prove both the approximation theorems stated in the end of Section 4.

Proof of Theorem 4. To find some relationship between the Wronskian $\det W_{n+1}$ and the determinant d_T from (15), we introduce an "intermediate" determinant

$$\tilde{d}(x, y) := \det \begin{pmatrix} h(x_0, y) & h_y(x_0, y) & \cdots & h_{y^{n-1}}(x_0, y) & h_{y^n}(x_0, y) \\ h_x(x_0, y) & h_{xy}(x_0, y) & \cdots & h_{xy^{n-1}}(x_0, y) & h_{xy^n}(x_0, y) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{x^{n-1}}(x_0, y) & h_{x^{n-1}y}(x_0, y) & \cdots & h_{x^{n-1}y^{n-1}}(x_0, y) & h_{x^{n-1}y^n}(x_0, y) \\ h_{x^n}(x, y) & h_{x^n y}(x, y) & \cdots & h_{x^n y^{n-1}}(x, y) & h_{x^n y^n}(x, y) \end{pmatrix}$$

For each fixed $y \in J$, Lemma 1 with $z_i = h_{y^{i-1}}(\cdot, y)$, $z = h_{y^n}(\cdot, y)$ leads to the conclusion that the estimate

$$\left|\frac{\tilde{d}(x, y)}{\det W_n(x_0, y)}\right| \leqslant \left\|\frac{\det W_{n+1}(\cdot, y)}{\det W_n(\cdot, y)}\right\| (1 + K_1 |x - x_0|) \tag{45}$$

holds for each $x \in I$ if K_1 is as in (34). Now for each fixed $x \in I$, we can apply Lemma 1 secondly, in this case with $z_j = h_{x^{j-1}}(x_0, \cdot)$, $z = h_{x^n}(x, \cdot)$ and conclude that the estimate

$$\left|\frac{d_T(x, y)}{\det W_n(x_0, y_0)}\right| \leqslant \left\|\frac{\tilde{d}(x, \cdot)}{\det W_n(x_0, \cdot)}\right\| (1 + K_2 |y - y_0|) \tag{46}$$

holds for each $y \in J$ if K_2 is as in (34). In view of ε-condition (5), it follows from (45) and (46) that the inequality

$$\left|\frac{d_T(x, y)}{\det W_n(x_0, y_0)}\right| \leqslant \varepsilon(1 + K_1 |x - x_0|)(1 + K_2 |y - y_0|)$$

holds at each point $(x, y) \in I \times J$. By Theorem 1, the last estimate yields (27), with α_T as indicated in (33). The proof is complete.

Proof of Theorem 5. Let us introduce another "intermediate" determinant

$$\tilde{d}(x, y) := \begin{pmatrix} h(x_1, y) & h_y(x_1, y) & \cdots & h_{y^{n-1}}(x_1, y) & h_{y^n}(x_1, y) \\ h(x_2, y) & h_y(x_2, y) & \cdots & h_{y^{n-1}}(x_2, y) & h_{y^n}(x_2, y) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(x_n, y) & h_y(x_n, y) & \cdots & h_{y^{n-1}}(x_n, y) & h_{y^n}(x_n, y) \\ h_{x^n}(x, y) & h_{x^n y}(x, y) & \cdots & h_{x^n y^{n-1}}(x, y) & h_{x^n y^n}(x, y) \end{pmatrix}$$

For each fixed $y \in J$, Lemma 2 with $z_i = h_{y^{i-1}}(\cdot, y)$, $z = h_{y^n}(\cdot, y)$ and $t_k = x_k$ leads to the conclusion that the estimate

$$\left|\frac{\tilde{d}(x, y)}{\det W_{x_1, \ldots, x_n}(y)}\right| \leqslant \left\|\frac{\det W_{n+1}(\cdot, y)}{\det W_n(\cdot, y)}\right\| (1 + K_3 l(x)) \tag{47}$$

holds for each $x \in I$ if K_3 is as in (37). Now for each fixed $x \in I$, we can apply Lemma 1 secondly, in this case with $z_j = h(x_j, \cdot)$, $z = h(x, \cdot)$ and $t_k = y_k$ to conclude that the estimate

$$\left|\frac{d_S(x, y)}{\det H}\right| \leqslant \left\|\frac{\tilde{d}(x, \cdot)}{\det W_{x_1, \ldots, x_n}(\cdot)}\right\| (1 + K_4 l(y)) \tag{48}$$

holds for each $y \in J$ if K_4 is as in (37). In view of ε-condition (5), it follows from (47) and (48) that the inequality

$$\left|\frac{d_S(x, y)}{\det H}\right| \leqslant \varepsilon(1 + K_3 l(x))(1 + K_4 l(y))$$

holds at each point $(x, y) \in I \times J$. By Theorem 2, the last estimate yields (28), with α_S as indicated in (36). The proof is complete.

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